THE FUGLEDE-KADISON DETERMINANT THEME AND VARIATIONS

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Abstract. We review the definition of determinants for finite von Neumann algebras, due to Fuglede and Kadison (1952), and a generalisation for appropriate groups of invertible elements in Banach algebras, from a paper by Skandalis and the author (1984). After some reminder on K-theory and Whitehead torsion, we hint at the relevance of these determinants to the study of L^2-torsion in topology. Table:
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1. **The classical setting**

1.1. **Determinants of matrices over commutative rings.** Let $\mathcal{R}$ be a ring with unit. For an integer $n \geq 1$, denote by $\mathrm{M}_n(\mathcal{R})$ the ring of n-by-n matrices over $\mathcal{R}$ and by $\mathrm{GL}_n(\mathcal{R})$ its group of units. $\mathcal{R}^*$ stands for $\mathrm{GL}_1(\mathcal{R})$.

Suppose $\mathcal{R}$ is commutative. The determinant

$$\det \, : \, \mathrm{M}_n(\mathcal{R}) \longrightarrow \mathcal{R} \tag{1}$$

is defined by a well-known explicit formula, polynomial in the matrix entries. It is alternate multilinear in the columns of the matrix, and normalised by $\det(1_n) = 1$; when $\mathcal{R}$ is a field, these properties constitute an equivalent definition, as was lectured by Weierstrass and Kronecker probably in the 1860's, and published much later (see [Kron–03,

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Siebzehnte Vorlesung, Page 291], published 12 years after Kronecker's death).

For $x, y \in \mathrm{M}_n(\mathcal{R})$, we have $\det(xy) = \det(x)\det(y)$. For $x \in \mathrm{M}_n(\mathcal{R})$ with $\det(x)$ invertible, an explicit formula shows that x itself is invertible, so that $\det(x) \in \mathcal{R}^*$ if and only if $x \in \mathrm{GL}_n(\mathcal{R})$. The restriction

$$\mathrm{GL}_n(\mathcal{R}) \longrightarrow \mathcal{R}^*, \qquad x \longmapsto \det x \tag{2}$$

is a group homomorphism.

1.2. **Three formulas for complex matrices involving determinants, exponentials, traces, and logarithms.** Suppose that $\mathcal{R}$ is *the field* $\mathbf{C}$ *of complex numbers.* The basic property of determinants that we wish to point out is the relation

$$\det(\exp y) \;=\; \exp(\mathrm{trace}(y)) \quad \text{for all } \; y \in \mathrm{M}_n(\mathbf{C}); \tag{3}$$

some exposition books give this as a very basic formula [Arno–73, § 16]; it will reappear below as Equation (21). It can also be written

$$\det(x) \;=\; \exp(\mathrm{trace}(\log x)) \quad \text{for appropriate } \; x \in \mathrm{GL}_n(\mathbf{C}). \tag{4}$$

"Appropriate" above can mean several things. If $\|x - 1\| < 1$, then $\log x$ can be defined by the convergent series

$$\log x \;=\; \log\big(1 + (x-1)\big)\,) \;=\; \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k. \tag{5}$$

If x is conjugate to a diagonal matrix, then $\log x$ can be defined component-wise (in pedantic terms, this is functional calculus, justified by the spectral theorem); in (4), note that the indeterminacy in the choice of the logarithm of a complex number is swallowed by the exponential, because $\exp 2\pi i = 1$.

Let $x \in \mathrm{GL}_n(\mathbf{C})$. Since the group is connected, we can choose a piecewise smooth path $\xi : [0,1] \longrightarrow \mathrm{GL}_n(\mathbf{C})$ from 1 to x. Since $\log \xi(\alpha)$ is a primitive of $\dot{\xi}(\alpha)\xi(\alpha)^{-1}d\alpha$, it follows from (4) that

$$\det(x) \overset{!}{=} \exp \int_0^1 \mathrm{trace}(\dot{\xi}(\alpha)\xi(\alpha)^{-1})d\alpha. \tag{6}$$

This will be our motivating formula for Section 6, and in particular for Equation (19).

The sign $\overset{!}{=}$ stands for a genuine equality, but indicates that some comment is in order. A priori, the integral depends on the choice of ξ, and we have also to worry about the determination of $\log \xi(\alpha)$. As there is *locally* no obstruction to choose a continuous determination of the primitive $\log \xi(\alpha)$ of $\dot{\xi}(\alpha)\xi(\alpha)^{-1}d\alpha$, the integral is invariant by small

changes of the path (with fixed ends), and therefore depends only on the homotopy class of ξ, so that it is defined modulo its values on (homotopy classes of) closed loops. The fundamental group $\pi_1(\mathrm{GL}_n(\mathbf{C}))$ is infinite cyclic, generated by the homotopy class of

$$\xi_0 \,:\, [0,1] \longrightarrow \mathrm{GL}_n(\mathbf{C}), \quad \alpha \longmapsto \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & 1_{n-1} \end{pmatrix},$$

and we have $\int_0^1 \mathrm{trace}(\dot{\xi_0}(\alpha)\xi_0(\alpha)^{-1})d\alpha = 2\pi i$. Consequently, the integral in the right-hand side of (6) is defined modulo $2\pi i\mathbf{Z}$, so that the right-hand side itself is well-defined. (This will be repeated in the proof of Lemma 10.)

Since a connected group is generated by any neighbourhood of the identity, there exist $x_1, \ldots, x_k \in \mathrm{GL}_n(\mathbf{C})$ such that $x = x_1 \cdots x_k$ and $\|x_j - 1\| < 1$ for $j = 1, \ldots, k$, and one can choose

$$\xi(\alpha) \;=\; \exp(\alpha(\log x_1)) \cdots \exp(\alpha(\log x_k)).$$

A short computation with this ξ gives

$$\exp \int_0^1 \mathrm{trace}(\dot{\xi}(\alpha)\xi(\alpha)^{-1})d\alpha \;=\; \exp(\mathrm{trace}(\log x_1)) \cdots \exp(\mathrm{trace}(\log x_k))$$

and it is now obvious that (4) implies (6).

1.3. **Historical note.** Determinants arise naturally with *linear systems of equations*, first with $\mathcal{R} = \mathbf{R}$, and more recently also with $\mathcal{R} = \mathbf{C}$. They have a prehistory in Chinese mathematics from the 2nd century BC [MacT]. In modern Europa, there has been an early contribution by[1] Leibniz in 1693, unpublished until 1850. Gabriel Cramer wrote an influential book, published in 1750. Major mathematicians who have written about determinants include Bézout, Vandermonde, Laplace, Lagrange, Gauss, Cauchy, Jacobi, Sylvester, Cayley, ... The connection between determinants of matrices in $M_3(\mathbf{R})$ and volumes of parallelepipeds is often attribued to Lagrange (1773). Let us mention an amazing book on the history of determinants [Muir–23]: four volumes, altogether more than 2000 pages, an ancestor of the *Mathematical Reviews*, for *one* subject, covering the period 1693–1900.

There is an extension of (1) to a skew-field k by Dieudonné, where the range of the mapping det defined on $\mathrm{M}(k)$ is[2] $(k^*/Dk^*) \sqcup \{0\}$ (see [Dieu–43], [Arti–57], and [Asla–96] for a discussion in case k is the skew-field of Hamilton quaternions). The extension of determinants for $\mathcal{R}$

[1] There are also resultants and determinants in the work of the Japanese mathematician Seki Takakazu, a contemporary of Leibniz and Newton.

[2] We denote by $D\Gamma$ the *group of commutators* of a group Γ.

a non-commutative ring has motivated a lot of work, in particular by Gelfand and co-autors since the early 1990's [GGRV–05]. Let us also mention a version for super-mathematics due to Berezin (see [Bere–79] and [Mani–88, Chapter 3]), as well as "quantum determinants", of interest in low-dimensional topology (see for example [HuLe–05]).

The notion of determinants extends to matrices over a ring without unit (by "adjoining a unit to the ring"). In particular, in functional analysis, there is a standard notion of determinants which appears in the theory of Fredholm integral equations, for example for operators on a Hilbert space of the form $1 + x$ where x is "trace-class" [Grot–56, Simo–79].

The oldest occurence I know of $\exp y$ or $\log x$, *including the notation*, defined by the familiar power series in the matrix y or $x - 1$, is in [Metz–92, Page 374]. But exponentials of linear differential operators appear also early in Lie theory, see for example [LieE–88, Page 75] and [Hawk–00, Page 82], even if Lie never uses a notation like $\exp X$ (unlike Poincaré, see his $e^{\alpha X}$ in [Poin–99, Page 177]).

There is a related and rather old formula known as the "Abel-Liouville-Jacobi-Ostrogradskii Identity". Consider a homogeneous linear differential equation of the first order $y'(t) = A(t)y(t)$, for an unknown function $y : [t_0, t_1] \longrightarrow \mathbf{R}^n$. The columns of a set of n linearly independent solutions constitute the *Wronskian matrix* $W(t)$. It is quite elementary (at least nowadays!) to show that $W'(t) = A(t)W(t)$, hence $\frac{d}{dt} \det W(t) = \mathrm{trace}(A(t)) \det W(t)$, and therefore

$$\det W(t) = \det W(t_0) \exp\Big(\int_{t_0}^{t} \mathrm{trace}(A(s))ds\Big),$$

a close cousin of Equation (6). The name of this identity refers to Abel (1827, case $n = 2$), Liouville and Ostrogradskii (1838), and Jacobi (1845). This was pointed out to me by Gerhard Wanner [HaNW–93, Section I.11]; also, Philippe Henry showed me this identity on the last but five line of [Darb–80].

Finally, two words about the authors of the 1952 paper alluded to in our title. *Bent Fuglede* is a Danish mathematician born in 1925. He has been working on mathematical analysis; he is also known for a book on *Harmonic maps between Riemannian polyhedra* (co-author Jim Eells, preface by Misha Gromov). *Richard Kadison* is an American mathematician, born in this same year 1925. He is known for his many contributions to operator algebras; his "global vision of the field was certainly essential for my own development" (words of Alain

Connes, when Kadison was awarded the Steele Prize 1999 for Lifetime Achievement, see [1999SP]).

1.4. **Plan.** Section 2 is a reminder on von Neumann algebras based on three types of examples, Section 3 is an exposition of the original Fuglede-Kadison idea, Section 4 stresses the difference between the complex-valued standard determinant and the real-valued Fuglede-Kadison determinant, Section 5 is a reminder on some notions of K-theory. Section 6 exposes the main variations of our title: determinants defined for connected groups of invertible elements in complex Banach algebras. We end by recalling in Section 7 a few facts about Whitehead torsion, with values in $\mathrm{Wh}(\Gamma)$, which is a quotient of the group K_1 of a group algebra $\mathbf{Z}[\Gamma]$, and by alluding in Section 8 at L^2-torsion, which is defined in terms of (a variant of) the Fuglede-Kadison determinant.

1.5. **Thanks.** I am grateful to Georges Skandalis for [HaS–84a], to Tatiana Nagnibeda and Stanislas Smirnov for their invitation to deliver a talk in Saint Petersburg on this subject, to Dick Kadison for encouragement to clean up my notes, as well as to Thierry Giordano, Jean-Claude Hausmann, Wolfgang Lück, Thierry Vust, and Claude Weber for useful comments.

2. On von Neumann algebras and traces

In a series of papers from 1936 to 1949, Murray and von Neumann founded the theory of *von Neumann algebras* (in their terminology *rings of operators*), which are complex $*$-algebras representable by unital weakly-closed $*$-algebras of some $\mathcal{L}(\mathcal{H})$, the algebra of all bounded operators on a complex Hilbert space $\mathcal{H}$.

We will first give three examples of pairs $(\mathcal{N}, \tau)$, with $\mathcal{N}$ a finite von Neumann algebra and τ a finite trace on it; we will then recall some general facts, and define a few terms, such as "finite von Neumann algebra", "finite trace", and "factor of type II_1".

Example 1 (**factors of type** I_n). *For any $n \geq 1$, the matrix algebra $\mathrm{M}_n(\mathbf{C})$ is a finite von Neumann algebra known as a* factor of type I_n. *The involution is given by $(x^*)_{j,k} = \overline{x_{k,j}}$. The linear form $x \longmapsto \frac{1}{n}\sum_{j=1}^{n} x_{j,j}$ is the (unique) normalised trace on $\mathrm{M}_n(\mathbf{C})$.*

Example 2 (**abelian von Neumann algebras**). *Let Z be a locally compact space and ν a positive Radon measure on Z. The space $L^\infty(Z,\nu)$ of complex-valued functions on Z which are measurable and ν-essentially bounded (modulo equality locally ν-almost everywhere) is an*

abelian von Neumann algebra. The involution is given by $f^*(z) = \overline{f(z)}$. *Any abelian von Neumann algebra is of this form.*

If ν *is a probability measure, the linear form* $\tau_\nu : f \longmapsto \int_Z f(z)d\nu(z)$ *is a trace on* $L^\infty(Z,\nu)$, *normalised in the sense* $\tau_\nu(1) = 1$.

Example 3 (**group von Neumann algebra**). *Let* Γ *be a group. The Hilbert space* $\ell^2(\Gamma)$ *has a scalar product, denoted by* $\langle\cdot|\cdot\rangle$, *and a canonical orthonormal basis* $(\delta_\gamma)_{\gamma\in\Gamma}$. *Consider the left-regular representation* λ *of* Γ *on* $\ell^2(\Gamma)$, *defined by* $(\lambda(\gamma)\xi)(x) = \xi(\gamma^{-1}x)$ *for all* $\gamma, x \in \Gamma$ *and* $\xi \in \ell^2(\Gamma)$.

The von Neumann algebra $\mathcal{N}(\Gamma)$ of Γ *is the weak closure in* $\mathcal{L}(\ell^2(\Gamma))$ *of the set of* $\mathbf{C}$*-linear combinations* $\sum_{\gamma\in\Gamma}^{\textit{finite}} z_\gamma\lambda(\gamma)$; *it is a finite von Neumann algebra. The involution is given by* $(z_\gamma\lambda(\gamma))^* = \overline{z_\gamma}\lambda(\gamma^{-1})$. *There is a canonical trace, given by* $x \longmapsto \langle x\delta_1|\delta_1\rangle$, *which extends* $\sum_{\gamma\in\Gamma}^{\textit{finite}} z_\gamma\lambda(\gamma) \longmapsto z_1$.

Moreover, $\mathcal{N}(\Gamma)$ *is a factor of type* II_1 *if and only if* Γ *is* icc[3] *(this is Lemma 5.3.4 in* [MuvN–43], *see also* [Dixm–57, chap. III, § 7, no 6]*).*

Remarks. (a) In the special case of a finite group, $\mathcal{N}(\Gamma)$ of Example 3 is a finite sum of matrix algebras as in Example 1. In the special case of an abelian group, $\mathcal{N}(\Gamma)$ of Example 3 is isomorphic, via Fourier transform, to the algebra of Example 2, with Z the Pontrjagin dual of Γ (a compact abelian group) and ν its normalised Haar measure.

(b) The von Neumann algebra $\mathcal{N}(\Gamma)$ is "of type I" if and only if Γ has an abelian subgroup of finite index [Thom–64]. It is "of type II_1" if and only if either[4] $[\Gamma : \Gamma_f] = \infty$, or $[\Gamma : \Gamma_f] < \infty$ and $|D\Gamma_f| = \infty$ [Kani–69]. There exist groups Γ such that $\mathcal{N}(\Gamma)$ is a non-trivial direct product of two two-sided ideals, one of type I and the other of type II_1; see [Kapl–51, Theorem 2] for the result, and [Newm–60] for explicit examples.

(c) Suppose in particular that Γ is finitely generated. If Γ_f is of finite index in Γ, then Γ_f is also finitely generated and it follows that Γ has an abelian subgroup of finite index. Thus $\mathcal{N}(\Gamma)$ is either of type I (if and only if Γ has a free abelian group of finite index) or of type II_1 (if and only if $[\Gamma : \Gamma_f] = \infty$); see [Kani–70].

(d) Other properties of $\mathcal{N}(\Gamma)$ are reviewed in [Harp–95].

[3]A group is *icc* if it is infinite and if all its conjugacy classes distinct from $\{1\}$ are infinite.

[4]We denote by Γ_f the union of the finite conjugacy classes of a group Γ. It is easy to check that Γ_f is a subgroup, and it is then obvious that it is a normal subgroup.

Let us now recall some general facts and some terminology, as announced.

(i) A von Neumann algebra $\mathcal{N}$ inherits *several natural topologies* from its representations by operators on Hilbert spaces, including the "ultraweak topology" (with respect to which the basic examples are separable) and the "operator topology" (with respect to which $\mathcal{N}$ is separable if and only if it is finite-dimensional).

(ii) There is available a *functional calculus*, justified by the *spectral theorem*: $f(x)$ is well-defined and satisfies natural properties, for $x \in \mathcal{N}$ normal ($x^*x = xx^*$) and f a complex-valued measurable function on the *spectrum*

$$\mathrm{sp}(x) \ := \ \{z \in \mathbf{C} \mid z - x \ \text{ is not invertible}\}$$

of x.

A von Neumann algebra $\mathcal{N}$ is *finite* if, for $x, y \in \mathcal{N}$, the relation $xy = 1$ implies $yx = 1$. A *projection* in a von Neumann algebra is a self-adjoint idempotent, in equations $e = e^* = e^2$. A von Neumann algebra $\mathcal{N}$ if *of type* I if, for any projection $0 \neq e \in \mathcal{N}$, there exists a projection $f \in \mathcal{N}$, $f \neq 0$ such that $fe = ef = f$ and $f\mathcal{N}f$ is abelian. A finite von Neumann algebra $\mathcal{N}$ if *of type* II_1 if, for any projection $0 \neq e \in \mathcal{N}$, the subset $e\mathcal{N}e$ is *not* abelian. It is known that any finite von Neumann algebra is the direct product of a finite algebra of type I and an algebra of type II_1.

(iii) A *finite trace* on a von Neumann algebra $\mathcal{N}$ is a linear functional $\tau : \mathcal{N} \longrightarrow \mathbf{C}$ which is continuous with respect to all the standard topologies on $\mathcal{N}$ (= which is *normal*, in the standard jargon), and which satisfies
  (iii_a) $\tau(x^*) = \overline{\tau(x)}$ for all $x \in \mathcal{N}$,
  (iii_b) $\tau(x^*x) \geq 0$, for all $x \in \mathcal{N}$,
  (iii_c) $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{N}$.
A trace is *faithful* if $\tau(x^*x) > 0$ whenever $x \neq 0$.

It is known that, on a finite von Neumann algebra which can be represented on a separable Hilbert space, there exists a faithful finite normal trace. Also, any linear form on $\mathcal{N}$ which is ultraweakly continuous and satisfies $(iii)_c$ can be written canonically as a linear combination of four linear forms satisfying the three conditions of (iii); this is a *Jordan decomposition* result of [Grot–57].

As we will not consider other kind of traces, we use "trace" for "finite trace" below.

A *factor* is a von Neumann algebra of which the centre coincides with the scalar multiples of the identity. A *factor of type* II_1 is an infinite dimensional finite factor; the discovery of such factors is one of the main results of Murray and von Neumann.

(iv) Let $\mathcal{N}$ be a factor of type II_1;
- $\circ$ $\mathcal{N}$ is a simple ring[5],
- $\circ$ there is a unique normalised[6] trace τ, which is faithful.

More precisely, on a factor $\mathcal{N}$ of type II_1, it is a standard result that there exists a unique normalised normal trace τ (in the sense of (iii) above); but unicity holds in a stronger sense, because any element in the kernel of τ is a finite sum of commutators [FaHa–80].

For a projection e, the number $\tau(e)$ is called the *von Neumann dimension* of e, or of the Hilbert space $e(\mathcal{H})$, when $\mathcal{N}$ is understood to be inside some $\mathcal{L}(\mathcal{H})$.

3. The Fuglede-Kadison determinant for finite von Neumann algebras

In 1952, Fuglede and Kadison defined their *determinant*

$$\det{}_\tau^{FK} \; : \; \begin{cases} \mathrm{GL}_1(\mathcal{N}) & \longrightarrow \qquad \mathbf{R}_+^* \\ \qquad x & \longmapsto \exp\Big(\tau\big(\log\big((x^*x)^{\frac{1}{2}}\big)\big)\Big) \end{cases} \tag{7}$$

which is a partial analogue of (2). The number $\det_\tau^{FK}(x)$ is well-defined by functional calculus, and most of the work in [FuKa–52] is for showing that $\det_\tau^{FK}$ is a homomorphism of groups. For the definition given below in Section 6, it will be the opposite: some work to show that the definition makes sense, but a very short proof to show it defines a group homomorphism.

In the original paper, $\mathcal{N}$ is a factor of type II_1, and τ is its unique trace with $\tau(1) = 1$; but everything carries over to the case of a von Neumann algebra and a normalised trace [Dixm–57, chap. I, § 6, no 11]. Besides being a group homomorphism, $\det_\tau^{FK}$ has the following

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[5]See [Dixm–57, chap. III § 5, no 2]. Words are often reluctant to migrate from one mathematical domain to another. Otherwise, one could define a factor of type II_1 as an infinite dimensional von Neumann algebra which is central simple. In the same vein, one could say that von Neumann algebras are principal rings; more precisely, in a von Neumann algebra $\mathcal{N}$, any ultraweakly closed left ideal is of the form $\mathcal{N}e$, where $e \in \mathcal{N}$ is a projection (this is a corollary of the von Neumann density theorem [Dixm–57, chap. I § 3, no 4]).

[6]The normalisation is most often by $\tau(1) = 1$. It can be otherwise, for example $\tau(1_n) = n$ on a factor of the form $\mathrm{M}_n(\mathcal{N})$, for some factor $\mathcal{N}$.

properties:

- $\det_\tau^{FK}(e^y) \,=\, |e^{\tau(y)}| \,=\, e^{\mathrm{Re}(\tau(y))}$ for all $y \in \mathcal{N}$
  and in particular $\det_\tau^{FK}(\lambda 1) \,=\, |\lambda|$ for all $\lambda \in \mathbf{C}$,
- $\det_\tau^{FK}(x) = \det_\tau^{FK}\big((x^*x)^{\frac{1}{2}}\big)$ for all $x \in \mathrm{GL}_1(\mathcal{N})$
  and in particular, $\det_\tau^{FK}(x) = 1$ for all $x \in \mathrm{U}_1(\mathcal{N})$.

For a $*$-ring $\mathcal{R}$ with unit, $\mathrm{U}_1(R)$ denotes its *unitary group*, defined to be $\{x \in R \mid x^*x = xx^* = 1\}$.

Instead of (7), we could equally view $\det_\tau^{FK}$ as a family of homomorphisms $\mathrm{GL}_n(\mathcal{N}) \longrightarrow \mathbf{R}_+^*$, one for each $n \geq 1$; if the traces on the $\mathrm{M}_n(\mathcal{N})$'s are normalised by $\tau(1_n) = n$, we have $\det_\tau^{FK}(\lambda 1_n) = |\lambda|^n$. More generally, for any projection $e \in \mathrm{M}_n(\mathcal{N})$, we have a von Neumann algebra $\mathrm{M}_e(\mathcal{N}) := e\,\mathrm{M}_n(\mathcal{N})e$, and a Fuglede-Kadison determinant $\det_\tau^{FK} : \mathrm{GL}_e(\mathcal{N}) \longrightarrow \mathbf{R}_+^*$ defined on its group of units.

There are extensions of $\det_\tau^{FK}$ to non-invertible elements, but this raises some problems, and technical difficulties. Two extensions are discussed in [FuKa–52]: the "algebraic extension" for which the determinant vanishes on singular elements (this is not mentionned anymore below), and the "analytic extension" which relies on Formula (7), in which one should understand

$$(8)\quad \det_\tau^{FK}(x) = \exp\Big(\tau\big(\log((x^*x)^{\frac{1}{2}})\big)\Big) = \exp\int_{\mathrm{sp}((x^*x)^{1/2})} \ln\lambda \, d\tau(E_\lambda),$$

where $(E_\lambda)_{\lambda\in\mathrm{sp}((x^*x)^{1/2})}$ holds for the spectral measure of $(x^*x)^{1/2}$; of course $\exp(-\infty) = 0$. (Note that we write "log" for logarithms of matrices and operators, and "ln" for logarithms of numbers.) For example, if x is such that there exists a projection e with $x = x(1-e)$ and $\tau(e) > 0$, we have $\det_\tau^{FK}(x) = 0$. For all $x, y \in \mathcal{M}$, we have

$$\begin{aligned} \det_\tau^{FK}((x^*x)^{1/2}) \;&=\; \lim_{\epsilon\to 0+} \det_\tau^{FK}((x^*x)^{1/2} + \epsilon 1) \\ \det_\tau^{FK}(x)\det_\tau^{FK}(y) \;&=\; \det_\tau^{FK}(xy) \end{aligned}$$

(see [FuKa–52], respectively Lemma 5 and Page 529). But an element x with $\det_\tau^{FK}(x) \neq 0$ need not be invertible, and no extension $\mathcal{M} \longrightarrow \mathbf{R}_+$ of the mapping $\det_\tau^{FK}$ of (7) is norm-continuous [FuKa–52, Theorem 6].

We will discuss another extension $\det_\tau^{FKL}$ to singular elements, in Section 8.

More generally, $\det_\tau^{FK}(x)$ can be defined for x an operator "affiliated" to $\mathcal{N}$, and also for traces which are *semi-finite* rather than finite as above. See [Grot–55, Arve–67, Fac–82b, Fack–83, Brow–86, HaSc–09],

among others. We will not have any further comment on this part of the theory.

Example 4 (**Fuglede-Kadison determinant for** $\mathrm{M}_n(\mathbf{C})$). *If* $\mathcal{N} = \mathrm{M}_n(\mathbf{C})$ *is the factor of type* I_n *(Example 1), if* det *denotes the* usual *determinant, and if* $\tau : x \longmapsto \frac{1}{n}\sum_{j=1}^{n} x_{j,j}$ *denotes the trace normalised by* $\tau(1_n) = 1$*, then*

$$\det{}_{\tau}^{FK}(x) \;=\; |\det(x)|^{1/n} \;=\; \det((x^*x)^{1/2})^{1/n} \tag{9}$$

for all $x \in \mathrm{M}_n(\mathbf{C})$.

Example 5 (**Fuglede-Kadison determinant for abelian von Neumann algebras**). *Let* $L^\infty(Z,\nu)$ *and* τ_ν *be as in Example 2, with* ν *a probability measure. The corresponding Fuglede-Kadison determinant is given by*

$$\det{}_{\tau}^{FK}(f) \;=\; \exp\int_Z \ln|f(z)|d\mu(z) \;\in\; \mathbf{R}_+. \tag{10}$$

In (10), observe that $\ln|f(z)|$ *is bounded above on* Z*, because* $|f(z)| \le \|f\|_\infty < \infty$ *for* ν*-almost all* z*. However* $|f(z)|$ *need not be bounded away from* 0*, so that* $\ln|f(z)| = -\infty$ *occurs. If the value of the integral is* $-\infty$*, then* $\det_{\tau}^{FK}(f) = \exp(-\infty) = 0$.

Consider an integer $d \ge 1$ and the von Neumann algebra $\mathcal{N}(\mathbf{Z}^d)$ of the free abelian group of rank d. Fourier transform provides an isomorphism of von Neumann algebras

$$\mathcal{N}(\mathbf{Z}^d) \xrightarrow{\approx} \mathrm{L}^\infty(T^d,\nu), \qquad x \longmapsto \hat{x},$$

where ν denotes the normalised Haar measure on the d-dimensional torus T^d. Moreover, the composition of this isomorphism with the trace τ_ν of Example 2 is the canonical trace on $\mathcal{N}(\mathbf{Z}^d)$, in the sense of Example 3.

Example 6 (**Fuglede-Kadison determinant and Mahler measure**). *Let* x *be a finite linear combination* $\sum_{n\in\mathbf{Z}^d}^{finite} z_n\lambda(n) \in \mathcal{N}(\mathbf{Z}^d)$*, so that* $\hat{x} \in L^\infty(T^d,\nu)$ *is a trigonometric polynomial. Then the* τ_ν*-Fuglede-Kadison determinant of* x *is given by the* exponential Mahler measure *of* $\hat{x}$*:*

$$\det{}_{\tau_\nu}^{FK}(x) \;=\; M(\hat{x}) \;:=\; \exp\int_{T^d} \ln|\hat{x}(z)|d\nu(z).$$

In the one-dimensional case ($d = 1$), if

$$\hat{x}(z) = a_0 + a_1 z + \cdots + a_s z^s = a_s\prod_{j=1}^{s}(z-\xi_j), \qquad \text{with } a_0 a_s \neq 0,$$

a computation shows that

$$\int_T \ln|\hat{x}(z)|d\nu(z) = \int_0^1 \ln|\hat{x}(e^{2\pi i\alpha})|d\alpha = \ln|a_s| + \sum_{j=1}^{s} \max\{1, |\xi_j|\}$$

(see [Schm–95, Proposition 16.1] or [Luck–02, Pages 135–7]).

Mahler measures occur in particular as entropies of $\mathbf{Z}^d$-actions by automorphisms of compact groups. More precisely, for $x \in \mathbf{Z}[\mathbf{Z}^d]$, which can be viewed as (the inverse Fourier transform of) a trigonometric polynomial, the group $\mathbf{Z}^d$ acts naturally on the quotient $\mathbf{Z}[\mathbf{Z}^d]/(x)$ of the group ring by the principal ideal (x), hence on the Pontryagin dual $\big(\mathbf{Z}[\mathbf{Z}^d]/(x)\big)\widehat{}$ of this countable abelian group, which is a compact abelian group. For example, if $x(z) = 1 + z - z^2 \in \mathbf{Z}[z, z^{-1}] \approx \mathbf{Z}[\mathbf{Z}]$, then $(\mathbf{Z}[\mathbf{Z}]/(x))\widehat{} \approx T^2$, and the corresponding action of the generator of $\mathbf{Z}$ on T^2 is described by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ [Schm–95, Example 5.3]. Every action of $\mathbf{Z}^d$ by automorphisms of a compact abelian group arises from some $x \in \mathbf{Z}[\mathbf{Z}^d]$ as above. More on this in [LiSW–90, Schm–95, Deni–06].

The logarithm of the Fuglede-Kadison determinant occurs also in the definition of "tree entropy", namely in the asymptotics of the number of spanning trees in large graphs [Lyon–05, Lyon–10].

4. **A motivating question**

It is natural to ask why $\mathbf{R}_+^*$ appears on the right-hand side of (7), even though $\mathcal{N}$ is a complex *algebra, for example a II_1-factor, whereas $\mathbf{C}^*$ appears on the right-hand side of (6) when $\mathcal{N} = \mathrm{M}_n(\mathbf{C})$.*

This is not due to some shortsightedness of Fuglede and Kadison. Indeed, for $\mathcal{N}$ a factor of type II_1, it has been shown that the Fuglede-Kadison determinant provides an *isomorphism* from the abelianised group $\mathrm{GL}_1(\mathcal{N})/D\,\mathrm{GL}_1(\mathcal{N})$ onto $\mathbf{R}_+^*$. In other words:

Proposition 7 (**Properties of operators with trivial Fuglede-Kadison determinant in a factor of type** II_1)**.** *Let $\mathcal{N}$ be a factor of type* II_1.

(i) *Any element in $\mathrm{U}_1(\mathcal{N})$ is a product of finitely many multiplicative commutators of unitary elements.*
(ii) *The kernel $\mathrm{SL}_1(\mathcal{N})$ of the homomorphism (7) coincides with the group of commutators of $\mathrm{GL}_1(\mathcal{N})$.*

Property (i) is due to Broise [Broi–67]. It is moreover known that any proper normal subgroup of $\mathrm{U}_1(\mathcal{N})$ is contained in its center, which

is $\{\lambda\,\mathrm{id} \mid \lambda \in \mathbf{C}^*, |\lambda| = 1\} \approx \mathbf{R}/\mathbf{Z}$ (see [Harp–79], Proposition 3 and its proof); this sharpens an earlier result on the classification of norm-closed normal subgroups of $\mathrm{U}_1(\mathcal{N})$ [Kadi–52, Theorem 2].

Property (ii) is [FaHa–80, Proposition 2.5]. It follows that the quotient of $\mathrm{SL}_1(\mathcal{N})$ by its center (which is the same as the centre of $\mathrm{U}_1(\mathcal{N})$) is simple, as an abstract group [Lans–70, Corollary 6.6, Page 123].

As a kind of answer to our motivating question, we will see below that, when the Fuglede-Kadison definition is adapted to a *separable* Banach algebra, the right-hand side of the homomorphism analogous to (7) is necessarily a quotient of the additive group $\mathbf{C}$ by a *countable* subgroup. For example, when $A = \mathrm{M}_n(\mathbf{C})$, this quotient is $\mathbf{C}/2i\pi\mathbf{Z} \overset{\exp(\cdot)}{\approx} \mathbf{C}^*$, see Corollary 14. On the contrary, when A is a II_1-factor (not separable as a Banach algebra), this quotient is $\mathbf{C}/2i\pi\mathbf{R} \overset{\exp(\mathrm{Re}(\cdot))}{\approx} \mathbf{R}_+^*$, see Corollary 15. The case of a separable Banach algebra can thus sometimes be seen as providing an interpolation between the two previous cases, see Remark 16.

5. A reminder on K_0, K_1, K_1^{top}, and Bott periodicity

5.1. **On $K_0(\mathcal{R})$ and $K_0(A)$.** Let $\mathcal{R}$ be a ring, say with unit to simplify several small technical points. Let us first recall one definition of the abelian group $K_0(\mathcal{R})$ of K-theory. We have a nested sequence of rings of matrices and (non-unital) ring homomorphisms

$$\begin{aligned} \mathcal{R} = \mathrm{M}_1(\mathcal{R}) \subset \cdots \subset \mathrm{M}_n(\mathcal{R}) \subset \mathrm{M}_{n+1}(\mathcal{R}) \subset \cdots \\ \subset \mathrm{M}_\infty(\mathcal{R}) := \bigcup_{n\geq 1} \mathrm{M}_n(\mathcal{R}), \end{aligned} \tag{11}$$

where the inclusions at finite stages are given by $x \longmapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$.

An *idempotent* in $\mathrm{M}_\infty(\mathcal{R})$ is an element e such that $e^2 = e$. Two idempotents $e, f \in \mathrm{M}_\infty(\mathcal{R})$ are *equivalent* if there exist $n \geq 1$ and $u \in \mathrm{GL}_n(\mathcal{R})$ such that $e, f \in \mathrm{M}_n(\mathcal{R})$ and $f = u^{-1}eu$. Define an *addition* on equivalence classes of idempotents, by

$$\begin{aligned} (\text{class of } e \in \mathrm{M}_k(\mathcal{R})) + (\text{class of } f \in \mathrm{M}_\ell(\mathcal{R})) \\ = \text{class of } e \oplus f \in \mathrm{M}_{k+\ell}(\mathcal{R}) \end{aligned} \tag{12}$$

where $e \oplus f$ denotes the matrix $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. Two idempotents $e, f \in \mathrm{M}_\infty(\mathcal{R})$ are *stably equivalent* if there exists an idempotent g such that the classes of $e \oplus g$ and $f \oplus g$ are equivalent; we denote by $[e]$ the stable equivalence class of an idempotent e. The set of stable equivalence

classes of idempotents, with the addition defined by $[e]+[f] := [e \oplus f]$, is the semi-group $K_0^+(\mathcal{R})$. The Grothendieck group $K_0(\mathcal{R})$ of this semi-group is the set of formal differences $[e]-[e']$, up to the equivalence defined by $[e]-[e'] \sim [f]-[f']$ if $[e]+[f'] = [e']+[f]$.

Note that K_0 is a functor: to any (unital) ring homomorphism $\mathcal{R} \longrightarrow \mathcal{R}'$ corresponds a natural homomorphism $K_0(\mathcal{R}) \longrightarrow K_0(\mathcal{R}')$ of abelian groups. Note also the isomorphism $K_0(\mathrm{M}_n(\mathcal{R})) \approx K_0(\mathcal{R})$, which is a straightforward consequence of the definition and of the isomorphisms $\mathrm{M}_k(\mathrm{M}_n(\mathcal{R})) \approx \mathrm{M}_{kn}(\mathcal{R})$.

(To an idempotent $e \in \mathrm{M}_\infty(\mathcal{R})$ is associated a $\mathcal{R}$-linear mapping $\mathcal{R}^n \to \mathcal{R}^n$ for n large enough, of which the image is a projective $\mathcal{R}$-module of finite rank. From this it can be checked that the definition of $K_0(\mathcal{R})$ given above coincides with another standard definition, in terms of projective modules of finite rank. Details in [Rose–94, Chap. 1].)

Rather than a general ring $\mathcal{R}$, consider now the case of a complex Banach algebra A with unit. For each $n \geq 1$, the matrix algebra $\mathrm{M}_n(A)$ is again a Banach algebra, for some appropriate norm, and we can furnish $\mathrm{M}_\infty(A)$ with the inductive limit topology. The following is rather easy to check, see e.g. [Blac–86, Pages 25–27]: two idempotents $e, f \in \mathrm{M}_\infty(A)$ are equivalent if and only if there exists a continuous path

$$[0,1] \longrightarrow \{\text{idempotents of } \mathrm{M}_\infty(A)\}, \qquad \alpha \longmapsto e_\alpha$$

such that $e_0 = e$ and $e_1 = f$. This has the following consequence:

Proposition 8. *If the Banach algebra A is separable, the abelian group $K_0(A)$ is countable.*

Proposition 9. *If $\mathcal{N}$ is a factor of type II_1, then $K_0(\mathcal{N}) \approx \mathbf{R}$ is uncountable.*

Indeed, if τ denotes the canonical trace on $\mathcal{N}$, the mapping which associates to the class of a self-adjoint idempotent e in $\mathcal{N}$ its von Neumann dimension $\tau(e) \in [0,1]$ extends to an isomorphism $K_0(\mathcal{N}) \xrightarrow{\approx} \mathbf{R}$.

On the proof : this follows from the "comparison of projections" in von Neumann algebras [Dixm–57, chap. III, § 2, no 7]. □

For historical indications on the early connections between K-theory and operator algebras, which goes back to the mid 60's, see [Rose–05].

5.2. **On $K_1(\mathcal{R})$.** For any ring $\mathcal{R}$ with unit, we have a nested sequence of group homomorphisms

$$\begin{aligned} \mathcal{R}^* = \mathrm{GL}_1(\mathcal{R}) \subset \cdots \subset \mathrm{GL}_n(\mathcal{R}) \subset \mathrm{GL}_{n+1}(\mathcal{R}) \subset \cdots \\ \subset \mathrm{GL}_\infty(\mathcal{R}) := \bigcup_{n \geq 1} \mathrm{GL}_n(\mathcal{R}), \end{aligned} \tag{13}$$

where the inclusions at finite stages are given by $x \longmapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$.

By definition,

$$K_1(\mathcal{R}) \ = \ \mathrm{GL}_\infty(\mathcal{R}) / D\,\mathrm{GL}_\infty(\mathcal{R}) \tag{14}$$

is an abelian group, usually written additively. Note that K_1 is a functor from rings to abelian groups.

For a commutative ring $\mathcal{R}$, the classical determinant provides a homomorphism $K_1(\mathcal{R}) \longrightarrow \mathcal{R}^*$; it is an isomorphism in several important cases, for example when $\mathcal{R}$ a field, or the ring of integers in a finite extension of $\mathbf{Q}$ [Miln–71, § 3]. In general ($\mathcal{R}$ commutative or not), the association of an element in $K_1(\mathcal{R})$ to a matrix in $\mathrm{GL}_\infty(\mathcal{R})$ can be viewed as a kind of determinant, or rather of a log of a determinant since $K_1(\mathcal{R})$ is written additively. Accordingly, the torsion defined in (24) below can be viewed as an alternated sum of log of determinants; we will remember this when defining the L^2-torsion in Equation (28).

Let $\mathcal{R}$ be again an arbitrary ring with unit. The *reduced* K_1-group is the quotient $\overline{K}_1(\mathcal{R})$ of $K_1(\mathcal{R})$ by the image of the natural homomorphism $\{1, -1\} \subset \mathrm{GL}_1(\mathcal{R}) \subset \mathrm{GL}_\infty(\mathcal{R}) \longrightarrow K_1(\mathcal{R})$.

In case $\mathcal{R} = \mathbf{Z}[\Gamma]$ is the integral group ring of a group Γ, the *Whitehead group* $\mathrm{Wh}(\Gamma)$ is the cokernel $K_1(\mathbf{Z}[\Gamma])/\langle \pm 1, \Gamma \rangle$ of the natural homomorphism $\Gamma \subset \mathrm{GL}_1(\mathbf{Z}[\Gamma]) \longrightarrow K_1(\mathbf{Z}[\Gamma]) \longrightarrow \overline{K}_1(\mathbf{Z}[\Gamma])$.

When Γ is finitely presented, there is a different (but equivalent) definition of $\mathrm{Wh}(\Gamma)$, with geometric content. In short, let L be a finite CW-complex with $\pi_1(L) = \Gamma$. One defines a group $\mathrm{Wh}(L)$ of appropriate equivalence classes of pairs (K, L), with K a finite CW-complex containing L in such a way that the inclusion $L \subset K$ is a homotopy equivalence. The unit is represented by pairs $L \subset K$ for which the inclusion is a *simple* homotopy equivalence. It can be shown that the functors $L \longrightarrow \mathrm{Wh}(L)$ and $L \longrightarrow \mathrm{Wh}(\pi_1(L))$ are naturally equivalent [Cohe–73, § 6 and Theorem 21.1].

Examples, for free abelian groups and free groups: $\mathrm{Wh}(\mathbf{Z}^d) = 0$ and $\mathrm{Wh}(F_d) = 0$ for $d \geq 0$. For finite cyclic groups, $\mathrm{Wh}(\mathbf{Z}/q\mathbf{Z})$ is a free abelian group of finite rank for all $q \geq 1$, and is zero if and only if $q \in \{1, 2, 3, 4, 6\}$.

From the standard references, let us quote [RhMK–67], [Miln–66], [Miln–71], [Cohe–73], and [Tura–01].

5.3. **On $K_1^{\mathrm{top}}(A)$, and on $K_0(A)$ viewed as a fundamental group.** Let A be a Banach algebra with unit. For each $n \geq 1$, the group $\mathrm{GL}_n(A)$ is an open subset of the Banach space $\mathrm{M}_n(A)$, and the induced topology makes it a topological group. The group $\mathrm{GL}_\infty(A)$ of (13) is also a topological group, for the inductive limit topology; we denote by $\mathrm{GL}^0_\infty(A)$ its connected component.

It is a simple consequence of the classical "Whitehead lemma" that, for any Banach algebra, the group $D\,\mathrm{GL}_\infty(A)$ is perfect and coincides with $D\,\mathrm{GL}^0_\infty(A)$; see for example [HaSk–85, Appendix]. In particular, $D\,\mathrm{GL}_\infty(A) \subset \mathrm{GL}^0_\infty(A)$, to that the quotient group

$$K_1^{\mathrm{top}}(A) \,:=\, \pi_0\left(\mathrm{GL}_\infty(A)\right) \,=\, \mathrm{GL}_\infty(A)/\,\mathrm{GL}^0_\infty(A) \tag{15}$$

is commutative. Note that $\mathrm{GL}_1(A)/\,\mathrm{GL}^0_1(A)$ need not be commutative [Yuen–73], even if its image in $\mathrm{GL}_\infty(A)/\,\mathrm{GL}^0_\infty(A)$ is always commutative.

Moreover, we have a natural quotient homomorphism

$$\mathrm{GL}_\infty(A)/D\,\mathrm{GL}^0_\infty(A) = K_1(A) \longrightarrow K_1^{\mathrm{top}}(A) = \mathrm{GL}_\infty(A)/\,\mathrm{GL}^0_\infty(A) \tag{16}$$

which is onto. It is an isomorphism if and only if the group $\mathrm{GL}^0_\infty(A)$ is perfect; this is the case if A is an infinite simple C*-algebra, for example if A is one of the Cuntz algebras O_n briefly mentionned below.

If the Banach algebra A is separable, the group $K_1^{\mathrm{top}}(A)$ is countable (compare with Proposition 8).

To an idempotent $e \in \mathrm{M}_n(A)$, we can associate the loop

$$\xi_e \,:\, \begin{cases} [0,1] & \longrightarrow \qquad\qquad \mathrm{GL}_n(A) \subset \mathrm{GL}_\infty(A) \\ \quad\alpha & \longmapsto \;\; \exp(2\pi i\alpha e) \,=\, \exp(2\pi i\alpha)e + (1-e); \end{cases} \tag{17}$$

note that $\xi_e(0) = \xi_e(1) = 1$. If two idempotents e and f have the same image in $K_0(A)$, it is easy to check that ξ_e and ξ_f are homotopic loops. It is a fundamental fact, which is a form of *Bott periodicity*, that the assignment $e \longmapsto \xi_e$ extends to a group isomorphism

$$K_0(A) \overset{\approx}{\longrightarrow} \pi_1\left(\mathrm{GL}^0_\infty(A)\right); \tag{18}$$

see [Karo–78, Theorem III.1.11] or [Blac–86, Chapter 9]. The terminology is due to a generalisation of (18): $K_i^{\mathrm{top}}(A) \approx K_{i+2}^{\mathrm{top}}(A)$ for any integer $i \geq 0$; by definition, $K_i^{\mathrm{top}}(A) = \pi_{i-1}\left(\mathrm{GL}_\infty(A)\right)$, for all $i \geq 1$, and $K_0^{\mathrm{top}}(A) = K_0(A)$.

5.4. **A few standard examples.** Let $A = \mathcal{C}(T)$ be the Banach algebra of continuous functions on a compact space T. Then $K_0(A) = K^0(T)$ and $K_1^{\mathrm{top}}(A) = K^1(T)$, where $K^0(T)$ and $K^1(T)$ stand for the (Grothendieck)-Atiyah-Hirzebruch-Bott K-theory groups of the topological space T, defined in terms of complex vector bundles. For example, if T is a sphere, we have

$$\begin{aligned} K_0(\mathcal{C}(\mathbf{S}^{2m})) &\approx \mathbf{Z}^2, & K_1^{\mathrm{top}}(\mathcal{C}(\mathbf{S}^{2m})) &= 0, \\ K_0(\mathcal{C}(\mathbf{S}^{2m+1})) &\approx \mathbf{Z}, & K_1^{\mathrm{top}}(\mathcal{C}(\mathbf{S}^{2m+1})) &\approx \mathbf{Z}, \end{aligned}$$

for all $m \geq 0$. If T is a compact CW-complex without cells of odd dimension, then $K_1^{\mathrm{top}}(\mathcal{C}(T)) = 0$.

Let A be an *AF-algebra*, namely a C*-algebra which contains a nested sequence $A_1 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots$ of finite-dimensional sub-C*-algebras with $\bigcup_{n\geq 1} A_n$ dense in A. Then $K_0(A)$ is rather well understood, and $K_1^{\mathrm{top}}(A) = 0$. The group $K_0(A)$ is the basic ingredient in Elliott's classification of AF-algebras, from the 1970's; this was the beginning of a long and rich story, with a numerous offspring, see [Blac–86, Chapter 7], [Rord–02], and [ElTo–08]. Here is a particular case, the so-called *CAR algebra*, or C*-algebra of the *Canonical Anticommutation Relations*, which is the C*-closure of the inductive sytem of finite matrix algebras

$$\mathbf{C} \subset \cdots \subset \mathrm{M}_{2^n}(\mathbf{C}) \subset \mathrm{M}_{2^{n+1}}(\mathbf{C}) \subset \cdots \subset \mathrm{M}_\infty(\mathbf{C})$$

where the inclusions at finite stages are given by $x \longmapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. For this,

$$K_0(CAR) = \mathbf{Z}[1/2] \quad \text{and} \quad K_1^{\mathrm{top}}(CAR) = 0$$

(for K_1 of CAR and a few other AF-algebras, see Subsection 6.1).

The Jiang-Su algebra $\mathcal{Z}$ is a simple infinite-dimensional C*-algebra (with unit) which has the same K-theory as $\mathbf{C}$ [JiSu–99]. It plays an important role in Elliott's classification program of C*-algebras.

The *reduced C*-algebra* of a group Γ is the *norm*-closure $C^*_\lambda(\Gamma)$ of the algebra $\left\{\sum_{\gamma\in\Gamma}^{\mathrm{finite}} z_\gamma \lambda(\gamma)\right\}$, see Example 3, in the algebra of all bounded operators on $\ell^2(\Gamma)$. For the free groups F_d (non-abelian free groups if $d \geq 2$), we have [PiVo–82]

$$K_0(C^*_\lambda(F_d)) \approx \mathbf{Z} \quad \text{and} \quad K_1^{\mathrm{top}}(C^*_\lambda(F_d)) \approx \mathbf{Z}^d.$$

For a so-called *irrational rotation C*-algebra* A_θ, generated by two unitaries u, v satisfying the relation $uv = e^{2\pi i\theta} vu$, where $\theta \in [0, 1]$ with $\theta \notin \mathbf{Q}$, we have [PiVo–80]

$$K_0(A_\theta) \approx \mathbf{Z}^2 \quad \text{and} \quad K_1^{\mathrm{top}}(A_\theta) \approx \mathbf{Z}^2.$$

For the infinite *Cuntz algebras* O_n, generated by $n \geq 2$ elements $s_1, \dots, s_n$ satisfying $s_j^* s_k = \delta_{j,k}$ and $\sum_{j=1}^n s_j s_j^* = 1$, we have [Cunt–81]

$$K_0(O_n) \approx \mathbf{Z}/(n-1)\mathbf{Z}, \quad \text{and} \quad K_1^{\mathrm{top}}(O_n) = 0.$$

For $\mathcal{N}$ a factor of type II_1, we have

$$K_0(\mathcal{N}) \approx \mathbf{R} \quad \text{and} \quad K_1(\mathcal{N}) = \mathbf{R}_+^*$$

(see Proposition 9 for K_0 and [LuRo–93] for K_1). More generally, for $\mathcal{N}$ a von Neumann algebra of type II_1, with centre denoted by $\mathcal{Z}$, we have

$$K_0(\mathcal{N}) \approx \{z \in \mathcal{Z} \mid z^* = z\},$$

where the right-hand side is viewed as a group for the addition, and

$$K_1(\mathcal{N}) \approx \{z \in \mathcal{Z} \mid z \geq \epsilon > 0\} \qquad (\epsilon \text{ depends on } z),$$

where the right-hand side is viewed as a group for the multiplication; see [Luck–02, Section 9.2]. For any von Neumann algebra $\mathcal{N}$

$$K_1^{\mathrm{top}}(\mathcal{N}) = 0,$$

because $\mathrm{GL}_n(\mathcal{N})$ is connected for all $n \geq 1$; indeed, by polar decomposition and functional calculus, any $x \in \mathrm{GL}_n(\mathcal{N})$ is of the form $\exp(a)\exp(ib)$, with a, b self-adjoint in $\mathrm{M}_n(\mathcal{N})$, so that x is connected to 1 by the path $\alpha \longmapsto \exp(\alpha a)\exp(i\alpha b)$.

5.5. **The topology of the group of units in a factor of type II_1, and Bott periodicity.** If $\mathcal{N}$ is a factor of type II_1, the isomorphism (18) of Bott periodicity shows that

$$\pi_1(\mathrm{GL}_\infty(\mathcal{N})) \approx K_0(\mathcal{N}) \approx \mathbf{R}.$$

More generally, by Bott periodicity,

$$\pi_{2j}(\mathrm{GL}_\infty(\mathcal{N})) = 0 \quad \text{and} \quad \pi_{2j+1}(\mathrm{GL}_\infty(\mathcal{N})) \approx \mathbf{R}$$

for all $j \geq 0$.

Moreover, it is known that $\pi_1(\mathrm{GL}_n(\mathcal{N})) \approx \mathbf{R}$, and that the embedding of $\mathrm{GL}_n(\mathcal{N})$ into $\mathrm{GL}_{n+1}(\mathcal{N})$ induces the identity on π_1, for all $n \geq 1$ [ArSS–71, Hand–78]. Note that, still for the norm topology, polar decomposition shows that the unitary group $\mathrm{U}_1(\mathcal{N})$ is a deformation retract of $\mathrm{GL}_1(\mathcal{N})$; in particular, we have also $\pi_1(\mathrm{U}_1(\mathcal{N})) \approx \mathbf{R}$.

For the strong topology, the situation is quite different; indeed, for "many" II_1-factors, for example for those associated to infinite amenable icc groups or to non-abelian free groups, it is known that the group $\mathrm{U}_1(\mathcal{N})^{\text{strong topology}}$ is contractible [PoTa–93].

6. **Revisiting the Fuglede-Kadison and other determinants**

Much of this section can be found in [HaS–84a]. For other expositions of part of what follows, see [CaFM–97, around Theorem 1.10] and [Luck–02, Section 3.2].

Let A be a complex Banach algebra (with unit, again for simplicity reasons), E a Banach space, and $\tau : A \longrightarrow E$ a continuous linear map which is *tracial*[7], namely such that $\tau(yx) = \tau(xy)$ for all $x, y \in A$. Then τ extends to a continuous linear map $\mathrm{M}_\infty(A) \longrightarrow E$, defined by $x \longmapsto \sum_{j \geq 1} \tau(x_{j,j})$, and again denoted by τ. If $e, f \in \mathrm{M}_\infty(A)$ are equivalent idempotents, we have $\tau(e) = \tau(f)$; it follows that τ induces a homomorphism of abelian groups

$$\underline{\tau} \, : \, K_0(A) \longrightarrow E, \qquad [e] \longmapsto \tau(e).$$

For example, if $A = \mathbf{C}$ and $\tau : \mathbf{C} \longrightarrow \mathbf{C}$ is the identity, the stable equivalence class of an idempotent $e \in \mathrm{M}_n(\mathbf{C})$ is precisely described by the dimension of the image $\mathrm{Im}(e) \subset \mathbf{C}^n$, so that $K_0(\mathbf{C}) \approx \mathbf{Z}$, and the image of $\underline{\tau}$ is the subgroup $\mathbf{Z}$ of the additive group $\mathbf{C}$.

For a piecewise differentiable path $\xi : [\alpha_1, \alpha_2] \longrightarrow \mathrm{GL}^0_\infty(A)$, we define

$$\begin{aligned} \widetilde{\Delta}_\tau(\xi) \; &= \; \frac{1}{2\pi i} \tau \Big(\int_{\alpha_1}^{\alpha_2} \dot{\xi}(\alpha) \xi(\alpha)^{-1} d\alpha \Big) \\ &= \; \frac{1}{2\pi i} \int_{\alpha_1}^{\alpha_2} \tau\big(\dot{\xi}(\alpha) \xi(\alpha)^{-1} \big) d\alpha. \end{aligned} \tag{19}$$

(If X is a compact space, for example if $X = [\alpha_1, \alpha_2] \subset \mathbf{R}$, the image of a continuous map $X \longrightarrow \mathrm{GL}^0_\infty(A)$ is inside $\mathrm{GL}_n(A)$, and therefore in the Banach space $\mathrm{M}_n(A)$, for n large enough; the integral can therefore be defined naively as a limit of Riemann sums.)

The normalisation in (19) is such that, if $A = \mathbf{C}$ and $\tau = \mathrm{id}$, the loop defined by $\xi_0(\alpha) = \exp(2\pi i \alpha)$ for $\alpha \in [0, 1]$ gives rise to $\widetilde{\Delta}_\tau(\xi_0) = 1$.

Lemme 10. *Let A be a complex Banach algebra with unit, E a Banach space, $\tau : A \longrightarrow E$ a tracial continuous linear map, and*

$$\widetilde{\Delta}_\tau \, : \, \{ \textit{paths in } \; \mathrm{GL}^0_\infty(A) \; \textit{ as above} \} \longrightarrow E$$

be the mapping defined by (19).

---

[7]If A is a C*-algebra, any such continuous linear form can be written canonically as a linear combination of four tracial continous linear forms, each of which being hermitian ($\tau(x^*) = \overline{\tau(x)}$) and positive ($\tau(x^* x) \geq 0$ for all $x \in A$). This is essentially due to Grothendieck ([Grot–57], already quoted in Section 2); see [CuPe–79, Proposition 2.7].

(i) *If ξ is the pointwise product of two paths ξ_1, ξ_2 from $[\alpha_1, \alpha_2]$ to $\mathrm{GL}^0_\infty(A)$, then $\widetilde{\Delta}_\tau(\xi) = \widetilde{\Delta}_\tau(\xi_1) + \widetilde{\Delta}_\tau(\xi_2)$.*

(ii) *If $\|\xi(\alpha) - 1\| < 1$ for all $\alpha \in [\alpha_1, \alpha_2]$, then $\tau\big(\dot{\xi}(\alpha)\xi(\alpha)^{-1}\big)d\alpha$ has a primitive $\tau\big(\log \xi(\alpha)\big)$, so that*

$$2\pi i \widetilde{\Delta}_\tau(\xi) = \tau(\log \xi(\alpha_2)) - \tau(\log \xi(\alpha_1)).$$

(iii) *$\widetilde{\Delta}_\tau(\xi)$ depends only on the homotopy class of ξ.*

(iv) *Let $e \in \mathrm{M}_\infty(A)$ be an idempotent and let ξ_e be the loop defined as in (17); then*

$$\widetilde{\Delta}_\tau(\xi_e) \;=\; \tau(e) \in E.$$

Sketch of proof. Claim (i) follows from the computation

$$\begin{aligned}
\widetilde{\Delta}_\tau(\xi_1\xi_2) \;&=\; \frac{1}{2\pi i}\int_{\alpha_1}^{\alpha_2} \tau\Big(\big(\dot{\xi_1}(\alpha)\xi_2(\alpha) + \xi_1(\alpha)\dot{\xi_2}(\alpha)\big)\xi_2(\alpha)^{-1}\xi_1(\alpha)^{-1}\Big)d\alpha \\
&= \frac{1}{2\pi i}\int_{\alpha_1}^{\alpha_2} \tau\big(\dot{\xi_1}(\alpha)\xi_1(\alpha)^{-1}\big)d\alpha \\
&\qquad\qquad + \frac{1}{2\pi i}\int_{\alpha_1}^{\alpha_2} \tau\big(\xi_1(\alpha)\dot{\xi_2}(\alpha)\xi_2(\alpha)^{-1}\xi_1(\alpha)^{-1}\big)d\alpha \\
&= \widetilde{\Delta}_\tau(\xi_1) \;+\; \widetilde{\Delta}_\tau(\xi_2).
\end{aligned}$$

Claims (ii) and (iii) are straightforward, compare with the end of Subsection 1.2. Claim (iv) follows again from an easy computation. □

Définition 11. *Let A be a complex Banach algebra with unit, E a Banach space, and $\tau : A \longrightarrow E$ a tracial continuous linear map. Define*

$$\Delta_\tau : \mathrm{GL}^0_\infty(A) \longrightarrow E/\underline{\tau}(K_0(A)) \tag{20}$$

to be the mapping which associates to an element x in the domain the class modulo $\underline{\tau}(K_0(A))$ of $\widetilde{\Delta}_\tau(\xi)$, where ξ is any piecewise differentiable path in $\mathrm{GL}^0_\infty(A)$ with origin 1 and extremity x.

Remark 12. *We insist on the fact that, in general, Δ_τ is not defined on the whole of $\mathrm{GL}_\infty(A)$.*

However, there are several classes of algebras for which is it known that the group $\mathrm{GL}_\infty(A)$ is connected. For example, this is the case for $A = \mathbf{C}$ (since $\mathrm{GL}_n(\mathbf{C})$ is connected for all $n \geq 1$) and for other finite-dimensional C-algebras (which are of the form $\prod_{j=1}^k \mathrm{M}_{n_j}(\mathbf{C})$), more generally for AF C*-algebras, and also for von Neumann algebras (viewed as Banach algebras).*

Theorem 13. *Let the notation be as above.*

(i) *The mapping Δ_τ of (20) is a homomorphism of groups, with image $\tau(A)/\underline{\tau}(K_0(A))$; in particular Δ_τ is onto if τ is onto.*
(ii) *$\Delta_\tau(e^y)$ is the class of $\tau(y)$ modulo $\underline{\tau}(K_0(A))$ for all $y \in \mathrm{M}_n(A)$.*

Corollary 14. *If $\tau : A \longrightarrow \mathbf{C}$ is such that $\underline{\tau}(K_0(A)) = \mathbf{Z}$, then*

$$\exp(2i\pi\Delta_\tau) : \mathrm{GL}^0_\infty(A) \longrightarrow \mathbf{C}^*$$

is a homomorphism of groups, and

$$\exp(2i\pi\Delta_\tau)(e^y) \;=\; e^{\tau(y)} \tag{21}$$

for all $y \in A$ (compare with (3)).

In particular, if $A = \mathbf{C}$ and if τ is the identity, then $\exp(2i\pi\Delta_\tau)$ is the usual determinant on $\mathrm{GL}_\infty(\mathbf{C})$.

Corollary 15. *If $\mathcal{N}$ is a factor of type II_1 and τ its canonical trace, then $\underline{\tau}(K_0(\mathcal{N})) = \mathbf{R}$,*

$$\exp(\mathrm{Re}(2i\pi\Delta_\tau)) : \mathrm{GL}_\infty(\mathcal{N}) \longrightarrow \mathbf{R}^*_+ \tag{22}$$

is a surjective homomorphism of groups, and its restriction to $\mathrm{GL}_1(\mathcal{N})$ is the Fuglede-Kadison determinant.

If A is a *separable* Banach algebra given with a trace τ, then the range of Δ_τ is the quotient of $\mathbf{C}$ by a *countable* group, by Proposition 8. Suppose more precisely that A is a C*-algebra with unit, that τ is a faithful tracial state on A which is *factorial*, and that the GNS-representation associated to τ provides an embedding $A \longrightarrow \mathcal{N}$ into a factor of type II_1, with τ on A being the restriction of the canonical trace on $\mathcal{N}$.

Remark 16. *Let $A, \tau, \mathcal{N}$ be as above. We have a commutative diagram*

$$\begin{array}{ccccccccc}
 & & \mathrm{GL}_\infty(\mathbf{C}) & \longrightarrow & \mathrm{GL}^0_\infty(A) & \longrightarrow & \mathrm{GL}_\infty(\mathcal{N}) & & \\
 & & \downarrow 2\pi i\Delta^{(\mathbf{C})}_\tau & & \downarrow 2\pi i\Delta^{(A)}_\tau & & \downarrow 2\pi i\Delta^{(\mathcal{N})}_\tau & & \\
\mathbf{C}^* & \approx & \mathbf{C}/2\pi i\mathbf{Z} & \longrightarrow & \mathbf{C}/2\pi i\underline{\tau}(K_0(A)) & \longrightarrow & \mathbf{C}/2\pi i\mathbf{R} & \approx & \mathbf{R}^*_+
\end{array}$$

where the top horizontal arrows are inclusions, the vertical homomorphisms $2\pi i\Delta$'s are onto, and the bottom horizontal arrows are onto.

In this sense, $\Delta^{(A)}_\tau$ can ve viewed as an interpolation between $\Delta^{(\mathbf{C})}_\tau$, which is essentially the classical determinant, and $\Delta^{(\mathcal{N})}_\tau$, which is essentially the Fuglede-Kadison determinant.

This situation occurs for example if $A = CAR$ (see 5.4); and also if $A = C^*_\lambda(\Gamma)$ is the reduced C*-algebra of an icc countable group Γ.

6.1. **On the sharpness of Δ's.** Let A be a complex Banach algebra. Denote by E_u the Banach space quotient of A by the closed linear span of the commutators $[x,y] = xy - xy$, $x, y \in A$; thus $E_u = A/\overline{[A,A]}$. The canonical projection $\tau_u : A \longrightarrow E_u$ is the *universal tracial continuous linear map* on A. In some cases, the space E_u has been characterised: for a finite von Neumann algebra $\mathcal{N}$ with centre $\mathcal{Z}$, the universal trace coincides with the canonical $\mathcal{Z}$-trace [Dixm–57, chap. III, § 5], and E_u can be identified with $\mathcal{Z}$ [FaHa–80, chap. 3]. Information on E_u for C*-algebras can be found in [CuPe–79] and [Fac–82a].

To the universal τ_u corresponds the *universal determinant*

$$\Delta_u : \mathrm{GL}^0_\infty(A) \longrightarrow E_u/\underline{\tau_u}(K_0(A)).$$

Observe that any tracial linear map $\tau : A \longrightarrow \mathbf{C}$ is the composition $\sigma\tau_u$ of the universal τ_u with a continuous linear form σ on E_u. We have

$$DGL^0_\infty(A) \overset{(1)}{\subset} \ker(\Delta_u) \overset{(2)}{\subset} \bigcap_{\sigma \in (E_u)^*} \ker(\Delta_{\sigma\tau_u}) \subset GL^0_\infty(A). \tag{23}$$

Both $\overset{(1)}{\subset}$ and $\overset{(2)}{\subset}$ can be strict inclusions, but $\overset{(2)}{\subset}$ is always an equality if A is separable. The last but one term on the right need not be closed in $GL^0_\infty(A)$. For all this, see [HaS–84a].

Let us agree that the universal determinant is *sharp* if the inclusions $\overset{(1)}{\subset}$ and $\overset{(2)}{\subset}$ are equalities, equivalently if the natural mapping from the kernel $\mathrm{GL}^0_\infty(A)/D\,\mathrm{GL}^0_\infty(A)$ of (16) to $E_u/\underline{\tau_u}(K_0(A))$ is an isomorphism.

If A is a simple AF C*-algebra with unit, its universal determinant is sharp. More precisely, if A is an AF-algebra with unit, $GL_n(A)$ is connected for all $n \geq 1$, and a fortiori so is $GL_\infty(A)$. If A is moreover simple, then

$$DGL_n(A) = \ker\left(\Delta_u : \mathrm{GL}_n(A) \longrightarrow E_u/\underline{\tau_u}(K_0(A))\right)$$

for all $n \geq 1$, and a similar equality holds for $\mathrm{U}_n(A)$ and $D\,\mathrm{U}_n(A)$ [HaS–84b, th. I and prop. 6.7]. If A is a simple C*-algebra with unit which is *infinite*, there are no traces on A [Fac–82a], and therefore no Δ_τ, and $GL^0_n(A)$ is a perfect group [HaS–84b, th. III].

Moreover, if G is one of these groups, the quotient of DG by its centre is a simple group [HaSk–85].

7. On Whitehead torsion

We follow [Miln–66].

7.1. **On K_1 and basis of modules.** Let $\mathcal{R}$ be a ring; we assume that free $\mathcal{R}$-modules of different finite ranks are not isomorphic. Let F be a free $\mathcal{R}$-module of finite rank, say n; let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be two basis of F. There is a matrix $x \in \mathrm{GL}_n(\mathcal{R})$ such that $a_j = \sum_{k=1}^n x_{j,k} b_k$, and therefore a class of x in $\overline{K}_1(\mathcal{R})$, denoted by $[b/a]$.

Let

$$C \,:\, 0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

be a chain complex of free $\mathcal{R}$-modules of finite ranks such that the homology groups H_i are also free $\mathcal{R}$-modules (the latter is automatic if $H_i = 0$, a case of interest in topology). Suppose that, for each i, there is given *a basis* c_i of C_i, and a basis h_i of H_i (the latter is automatic if $H_i = 0$).

In a first time, assume that each boundary submodule B_i is also free, with a basis b_i. Using the inclusions $0 \subset B_i \subset Z_i \subset C_i$ and the isomorphism $Z_i/B_i \approx H_i$, $C_i/Z_i \approx B_{i-1}$, there is a natural way to define (up to some choices) *a second basis* of C_i denoted by $b_i h_i b_{i-1}$. By definition, the *torsion* of C, given together with the basis c_i and h_i, is the element[8]

$$\tau(C) \,=\, \sum_{i=0}^{n} (-1)^i [b_i h_i b_{i-1}/c_i] \,\in\, \overline{K}_1(\mathcal{R}). \tag{24}$$

It can be shown to be independent of the other choices made to define $b_i h_i b_{i-1}$; in particular, the signs $(-1)^i$ are crucial for $\tau(C)$ to be independent of the choices of the basis b_i's.

In case the hypothesis on B_i being free is not fulfilled, it is easy to check that the B_i's are stably free, and there is a natural way to extend the definition of $\tau(C)$. (A $\mathcal{R}$-module A is *stably free* if there exist a free $\mathcal{R}$-module F such that $A \oplus F$ is free.) This can be read in [Miln–66, § 1-6].

Suppose now that C is *acyclic*, namely that $H_*(C) = 0$. There exists a *chain contraction*, namely a degree one morphism $\delta : C \longrightarrow C$ such that $\delta d + d\delta = 1$, and therefore an isomorphism

$$d + \delta|_{\mathrm{odd}} \,:\, C_{\mathrm{odd}} = C_1 \oplus C_3 \oplus \cdots \longrightarrow C_{\mathrm{even}} = C_0 \oplus C_2 \oplus \cdots . \tag{25}$$

[8]The occurence of the same letter τ for the torsion here and for traces above has no other reason than standard use.

Since C_{odd} and C_{even} have basis (from the c_i 's), this isomorphism defines an element in $\overline{K}_1(\mathcal{R})$; we have

$$\tau(C) \;=\; \text{class of } d + \delta|_{\text{odd}} \text{ in } \overline{K}_1(\mathcal{R}) \tag{26}$$

(see [Cohe–73, Chap. III]). Formula (26) is sometimes better suited than (24).

7.2. **The Whitehead torsion of a pair** (K, L)**.** Consider a pair (K, L) of a finite connected CW-complex K and a subcomplex L which is a deformation retract of K; set $\Gamma = \pi_1(L) \approx \pi_1(K)$. For a CW-pair (X, Y), consider the complex which defines cellular homology theory, with groups $C_i^{\text{CW}}(K, L) = H_i^{\text{sing}}(|X^i \cup Y|, |X^{i-1} \cup Y|)$; here, H_i^{sing} denotes singular homology with trivial coefficients $\mathbf{Z}$, and $|X^i \cup Y|$ denotes the space underlying the union of the ith skeleton of X with Y. If $\tilde{K}$ and $\tilde{L}$ denote the universal covers of L and K, the groups $C_i^{\text{CW}}(\tilde{K}, \tilde{L})$ are naturally free $\mathbf{Z}[\Gamma]$-modules; moreover, they have free basis as soon as a choice has been made of one oriented cell in $\tilde{K}$ above each oriented cell in K. For each of these choices, and the corresponding basis, we have a torsion element $\tau(C^{\text{CW}}(K, L)^{+\text{choices}}) \in \overline{K}_1(\mathbf{Z}[\Gamma])$. To obtain an element independent of these choices, it suffices to consider the quotient $\text{Wh}(\Gamma) = K_1(\mathbf{Z}[\Gamma])/\langle \{1, -1\}, \Gamma \rangle$ defined in 5.2. The class

$$\tau(K, L) \;\in\; \text{Wh}(\Gamma)$$

of $\tau(C^{\text{CW}}(K, L)^{+\text{choices}})$ is the *Whitehead torsion* of the pair (K, L). In 1966, it was known to be *combinatorially invariant* (namely invariant by subdivision of CW-pairs); more on this in [Miln–66, § 7]. Since then, it has been shown to be a *topological invariant* of the pair $(|K|, |L|)$ [Chap–74]; this was a spectacular success of infinite dimensional topology.

7.3. **On torsion and cobordism.** A *h-cobordism* is a triad $(W; M, M')$ where W is a smooth manifold whose boundary is the disjoint union $M \sqcup M'$ of two closed submanifolds, such that both M and M' are deformation retracts of W. Products $W = M \times [0, 1]$ provide trivial examples; in [Miln–61], there is a non-trivial example of h-cobordism $(W, L \times \mathbf{S}^4, L' \times \mathbf{S}^4)$, with L and L' two 3-dimensional lens manifolds[9] which are homotopically equivalent but not homeomorphic. More generally, by a 1965 result of Stallings [Miln–66, § 11]:

> *If* $\dim M \geq 5$*, any* $\tau \in \text{Wh}(\pi_1(M))$ *is of the form* $\tau(W, M)$ *for some h-cobordism* $(W; M, M')$*. Moreover, for two cobordisms*

[9] More precisely, L and L' are quotients of $\mathbf{S}^3$ by free actions of $\mathbf{Z}/7\mathbf{Z}$.

$(W_1; M, M_1)$, $(W_2; M, M_2)$ such that $\tau(W_1, M) = \tau(W_2, M)$, there exist a diffeomorphism $W_1 \longrightarrow W_2$ which preserves M.

A h-cobordism gives rise to a chain complex and a torsion invariant $\tau(W, M) \in \mathrm{Wh}(\Gamma)$, as in 7.2. Here is the basic *s-cobordism theorem* of Barden, Mazur, and Stallings [Kerv–65]:

If $\dim W \geq 6$, then W is diffeomorphic to the product $M \times [0,1]$ if and only if $\tau(W, M) = 0 \in \mathrm{Wh}(\Gamma)$.

In particular, if $\pi_1(M) = \{1\}$, then W is always diffeomorphic to $M \times [0,1]$; this is the *h-cobordism theorem* of [Smal–62].

For example, if Σ is a homotopy sphere of dimension $n \geq 6$, if W is the complement in Σ of two disjoint open discs, and if S_0, S_1 are the boundaries of these discs (they are standard spheres), then $(W; S_0, S_1)$ is a h-cobordism, and W is diffeomorphic to $S^{n-1} \times [0,1]$. It follows that Σ is *diffeomorphic* to a manifold obtained by gluing together the boundaries of two closed n-balls under a suitable diffeomorphism, and that Σ is *homeomorphic* to the standard n-sphere; the last conclusion extends to the case of dimension $n = 5$. This is the generalised Poincaré conjecture in large dimensions, established in the early 60's. The first proof was that of Smale (see [Smal–61], and slightly later [Smal–62]); very soon after, there has been other proofs of other formulations[10] of the Poincaré conjecture, logically independent of Smale's proof but inspired by his work, by Stallings (for $n \geq 7$) and Zeeman (for $n \geq 5$). The other dimensions were settled much later: by Freedman in 1982 for $n = 4$ and by Perelman in 2003 for $n = 3$.

7.4. **On the Reidemeister-Franz-de Rham torsion.** Since K_1 is a functor, any linear representation $h : \Gamma \longrightarrow \mathrm{GL}_n(\mathbf{R})$ provides a ring homomorphism $\mathbf{Z}[\Gamma] \longrightarrow \mathrm{M}_n(\mathbf{R})$, therefore a morphism of abelian group

$$K_1(\mathbf{Z}[\Gamma]) \longrightarrow K_1(\mathrm{M}_n(\mathbf{R})) = K_1(\mathbf{R}) \approx \mathbf{R}^*,$$

where $\approx$ is induced by the determinant $\mathrm{GL}_\infty(\mathbf{R}) \longrightarrow \mathbf{R}^*$, and also a morphism $\overline{K}_1(\mathbf{Z}[\Gamma]) \longrightarrow \overline{K}_1(\mathbf{R}) \approx \mathbf{R}^*_+$. When the representation is orthogonal, $h : \Gamma \longrightarrow O(n)$, this induces a morphism of abelian groups $\mathrm{Wh}(\Gamma) \longrightarrow \mathbf{R}^*_+$.

For a complex of $\mathbf{Z}[\Gamma]$-modules C with Whitehead torsion $\tau(C) \in \mathrm{Wh}(\Gamma)$, the image of $\tau(C)$ is the *Reidemeister torsion* $\tau_h(C) \in \mathbf{R}^*_+$, which is a real number (in fact, $\tau_h(C)$ may be well-defined even in cases $\tau(C)$ is not ...). This is the basic invariant in important work by Reidemeister, Franz, and de Rham (earliest papers pulished in 1935).

[10]For a description of the various formulations, written for non-specialists, see [Miln–11].

8. A few lines on L^2-torsion

8.1. **Another extension of $\det_\tau^{FK}$.** Let $\mathcal{N}$ be a finite von Neumann algebra and let $\tau : \mathcal{N} \longrightarrow \mathbf{C}$ be a finite trace. For $x \in \mathcal{N}$, let $(E_\lambda)_{\lambda \geq 0}$ denote the spectral measure of $(x^*x)^{\frac{1}{2}}$. Define
(27)

$$\det_\tau^{FKL}(x) \; = \; \begin{cases} \exp \lim_{\epsilon \to 0+} \int_\epsilon^\infty \ln(\lambda) \, d\tau(E_\lambda) & \text{if } \lim_{\epsilon \to 0+} \int_\epsilon^\infty \cdots > -\infty, \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate that $\det_\tau^{FKL}(x) = \det_\tau^{FK}(x)$ when x is invertible, but the equality does not hold in general ($\det_\tau^{FK}(x)$ is as in (8)). For example, if $x \in \mathrm{GL}_1(\mathcal{M})$, and $X = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathcal{M})$, we have

$$0 \; = \; \det_\tau^{FK}(X) \; \neq \; \det_\tau^{FKL}(X) \; = \; \det_\tau^{FKL}(x) \; = \; \det_\tau^{FK}(x) \; > \; 0.$$

The main properties of $\det_\tau^{FKL}$, including

$$\begin{aligned} &\det_\tau^{FKL}(xy) \; = \; \det_\tau^{FKL}(x)\det_\tau^{FKL}(y) \;\text{ for }\; x, y \in \mathcal{M} \\ &\text{such that }\; y \;\text{ is injective and }\; x \;\text{ has dense image} \end{aligned}$$

are given in [Luck–02, Theorem 3.14].

8.2. **On L^2-torsion.** Since Atiyah's work on the L^2-index theorem [Atiy–76], we know that (complexes of) $\mathcal{N}$-modules are relevant in topology, say for $\mathcal{N} = \mathcal{N}(\Gamma)$ and for Γ the fundamental group of the relevant space. Let $\mathcal{N}$ and τ be as above. Let

$$C \; : \; 0 \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

be a finite complex of $\mathcal{N}$-modules, with appropriate finiteness conditions on the modules (they should be projective of finite type), with a condition of acyclicity on the homology (the image of d_j should be dense in the kernel of d_{j-1} for all j), and with a non-degeneracy condition on the differentials d_j (which should be of "determinant class", namely $\det_\tau^{FKL}(d_j^* d_j)$ should be as in the first case of (27)). The L^2-*torsion* of C is defined to be

$$\rho^{(2)}(C) \; = \; \sum_k (-1)^k \ln \det_\tau^{FKL}((d_j^* d_j)^{1/2}) \; \in \; \{-\infty\} \sqcup \mathbf{R}. \tag{28}$$

There is a L^2-analogue of (26), see [Luck–02, 3.3.2].

L^2-torsion, and related notions, have properties which parallel those of classical torsions, in particular of Whitehead torsion, and seem to be relevant for geometric problems, e.g. for understanding volumes

of hyperbolic manifolds of odd dimensions. We refer (once more) to [Luck–02].

8.3. **Speculation.** It is tempting to ask wether (or even speculate that!) modules over reduced C*-algebras $A = C^*_{red}(\Gamma)$ and refinements $\Delta_\tau^{(A)}$ will be relevant one time or another, rather than modules over $\mathcal{N}(\Gamma)$ and Fuglede-Kadison determinants $\det_\tau^{FK}(\cdot)$. Compare with Remark 16.

Technical problems involving extensions of these "determinants" $\Delta_\tau^{(A)}$ to singular elements are likely to occur.

?!??!!???!!! !!!???!!??!?

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